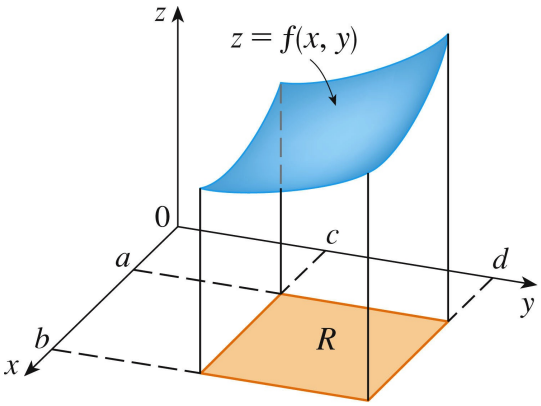
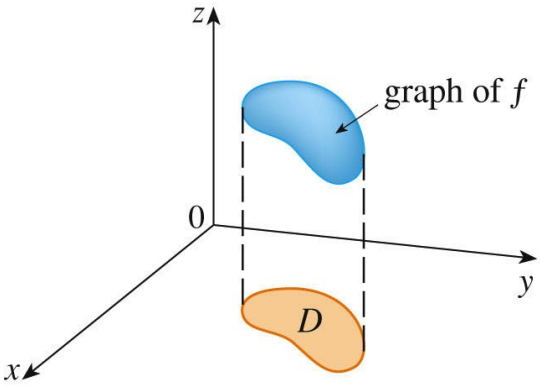
Lesson 23. Double Integrals Over General Regions

1 In the previous lesson: rectangles



- Rectangle $R = [a,b] \times [c,d]$
  $= \{(x,y) \mid a \le x \le b, c \le y \le d\}$
- $$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx$$
  $$= \int_c^d \int_a^b f(x,y)\,dx\,dy$$

2 General regions

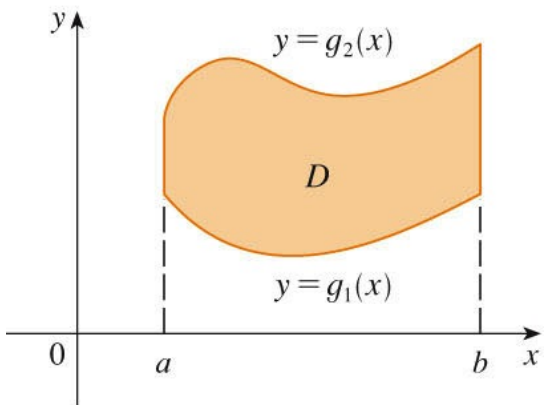


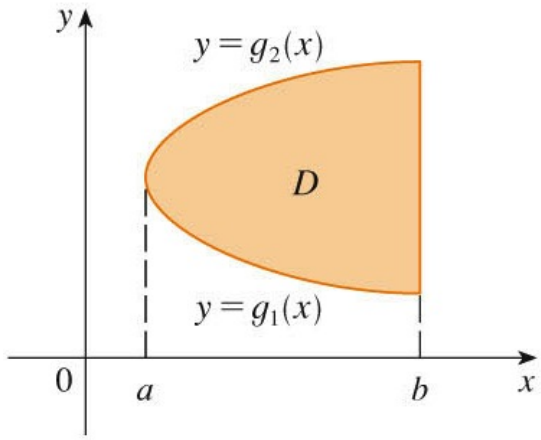
- How about general regions D?
- Intuition: if $f(x,y) \ge 0$, double integral still represents volume of solid between D and graph of f
- We focus on <u>two</u> types of regions

- **Type I regions**: lies between the graphs of two continuous functions of <u>x</u>, that is:

$$D = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$$

  - Examples:





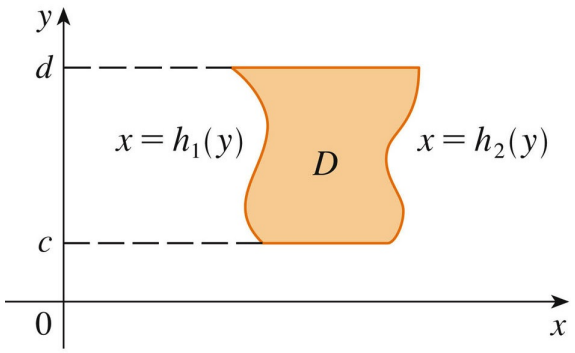
- If D is a type I region (and f is continuous on D), then

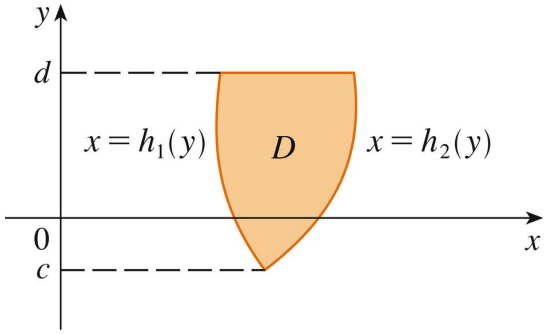
- In the inner integral, x is regarded a constant in $f(x, y)$ and the limits of integration

- **Type II regions**: lies between the graphs of two continuous functions of y, that is:

$$D = \{(x, y) \mid c \le x \le d,\ h_1(y) \le x \le h_2(x)\}$$

- Examples:





- If D is a type II region (and f is continuous on D), then

- In the inner integral, y is regarded a constant in $f(x, y)$ and the limits of integration

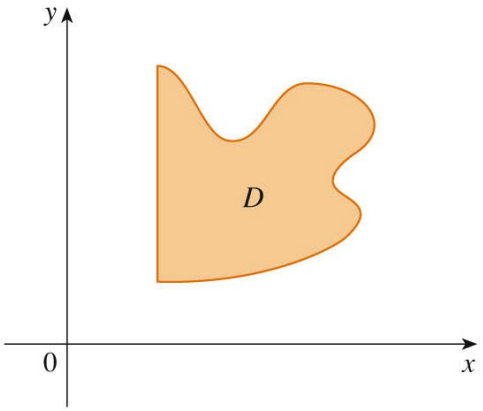
Example 1. Find $\iint_D (x - y)\, dy\, dx$, where $D = \{(x, y) \mid 0 \le x \le 1, 2x \le y \le 2\}$.

Example 2. Find $\iint_D (x+2)\,dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

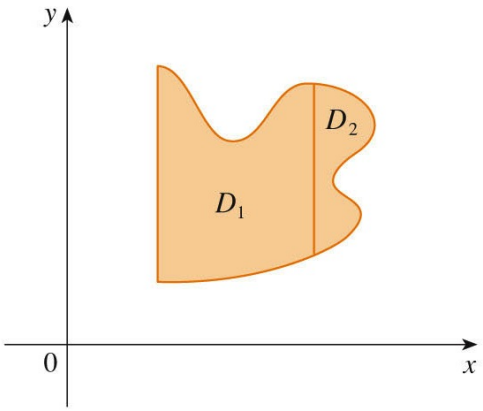
Example 3. Evaluate $\iint_D (2 - 2x - y)\,dA$, where D is the triangular region with vertices $(0,0)$, $(2,0)$, $(1,1)$.

- If $D = D_1 \cup D_2$, where D_1 and D_2 don't overlap except perhaps on their boundaries, then

$$\iint_D f(x, y)\, dA = \iint_{D_1} f(x, y)\, dA + \iint_{D_2} f(x, y)\, dA$$



(a) D is neither type I nor type II.



(b) $D = D_1 \cup D_2$, D_1 is type I, D_2 is type II.

Example 4. Write $\iint_D (2 - 2x - y)\, dA$ as the sum of 2 type I region iterated integrals, where D is the triangular region with vertices $(0, 0)$, $(2, 0)$, and $(1, 1)$.

Example 5. Sketch the region of integration of $\int_0^2 \int_{x^2}^{2x} (x^2 + y^2)\, dy\, dx$. Change the order of integration. Evaluate the integral using the order of integration of your choice.

Example 6. Consider the double integral $\iint_D f(x, y)\, dA$ where D is enclosed by $x = 0$, $x = \sqrt{1 - y^2}$. Set up this double integral as an iterated integral using both orders of integration.

Example 7. Consider the double integral $\int_0^4 \int_{\sqrt{x}}^2 f(x,y)\,dy\,dx$. Sketch the region of integration and change the order of integration.

Example 8. Let D be some region in the xy-plane. What does $\iint_D 1\,dA$ represent? Explain.